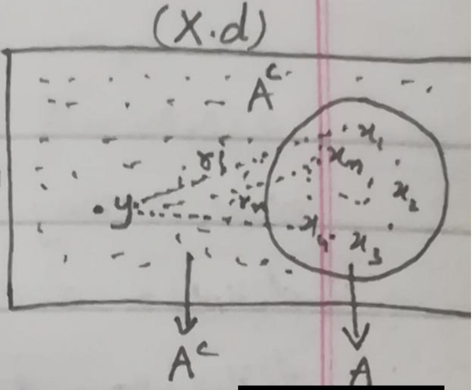
<u>Theorem</u>; The compliment of finite set in a metric space is open set;



<u>Proof</u> Let $A = \{x_1, x_2, \dots x_n\} = \{x_i\}_{i=1}^{n}$ be a finite subset of "X"

$A \subseteq X$

Let $y \in A^c$, $y \notin A$

$y \neq x_i \ (x_1, x_2, \dots x_n) \quad i = 1, 2, \dots n$

$d(x_i, y) \neq 0$

Let $d(x_i, y) = r_i \ \forall \ i = 1, 2, 3, \dots n$ and also

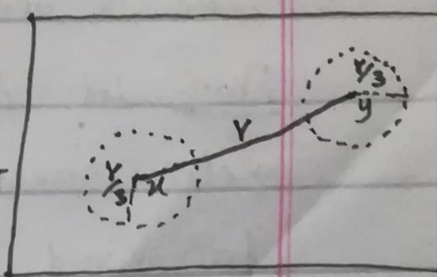
Let $r_i = \min\{r_1, r_2 \dots r_n\} = \{r_i\}_{i=1}^{n}$

Let an open sphere of radius "r" with center at "y" does not contain an point of set "A" so,

$S_r(y) \cap A = \phi$ and $A^c \cap A = \phi$ so, $S_r(y) \subseteq A^c$

it holds for all $y \in A^c$, A^c is an open set in $(X.d)$

<u>Housdorff property</u> every two distinct points in a metric space are separated by pair of disjoint open sphere containing the respective points.



<u>Proof</u> let $(X.d)$ be an arbitrary metric space containing two distinct points. x and y such that

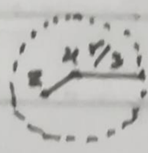
$d(x, y) = r$, Now taking two open sphere at x and y of radius : $\frac{x}{3}$:

we are to prove that $S_{r/3}(x) \cap S_{r/3}(y) = \phi$

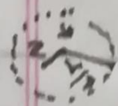
Let if $S_{r/3}(x) \cap S_{r/3}(y) \neq \phi$ There exist z point

$z \in S_{r/3}(x) \cap S_{r/3}(y)$

$z \in S_{r/3}(x)$, $z \in S_{r/3}(y)$



$d(x,z) < r/3$, $d(y,z) < r/3$



Now appling Triangular inequality on x, y, z

$d(x,y) \leq d(x,z) + d(y,z)$

$d(x,y) < \frac{r}{3} + \frac{r}{3}$

$r < \frac{r+r}{3}$, $r < \frac{2r}{3}$, $1 < \frac{2}{3}$ is false

So, our sopposition $S_{\frac{r}{3}}(x) \cap S_{\frac{r}{3}}(y) \neq \phi$ is wrong

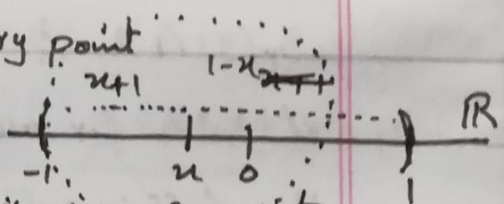
Hence $S_{\frac{r}{3}}(x) \cap S_{r/3}(y) = \phi$. x, y distinct point open sphere are dijaint pair.

Theorem Any open interval on $\mathbb{R}$ is an open set

Proof Let $(-1,1) \subseteq \mathbb{R}$ be an open interval in $\mathbb{R}$.

$x \in (-1,1)$ be its an arbitrary point

Taking $r = \min\{|x+1|, |1-x|\}$



$S_r(x) \subseteq (-1,1)$, so, $(-1,1)$ is an open set

Theorem every Non-empty subset in a discrete metric space is open set. $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

Proof Let $A \neq \phi$ bet a subset of $\mathbb{R}$ (x,d) a discrete metric space

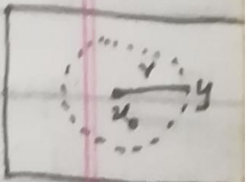
for $r \geq 1$, $S_r(x) = A = X$ $\forall x \in X = A$

for $r < 1$ $S_r(x) = \{x\}$ $\forall x \in A = X$

from $r \geq 1$ and $r < 1$ is a openset in metric space.

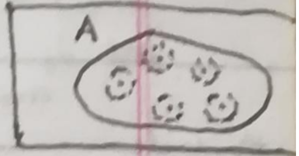
Land Mark Theorem for open sets (Arbitrary Union)

Arbitrary union of open set in a metric space is open set.



Proof open sphere, $S_r(x) = \{y \in x : d(x,y) < r\}$

open set 'A' is open in $(X.d)$ if all of its centre of some open sphere

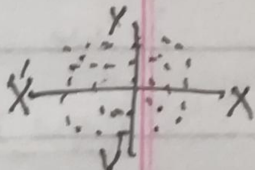


Arbitrary Union

finite — countable — infinit — Uncountable

$A_1 \cup A_2 \cup A_3 \ldots A_n$ $\quad$ $A_1 \cup A_2 \cup A_3 \ldots = \bigcup_{i=1}^{\infty} A_i$ $\quad$ $\bigcup_{\alpha \in I} A_\alpha$

e.g **uncountable** $A_x = \{(x, y) : y \in \mathbb{R}\}$

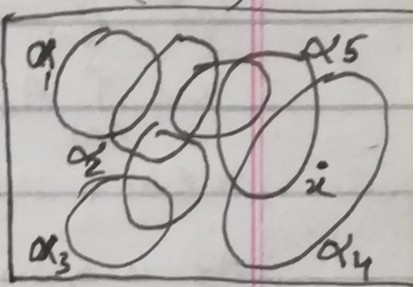


$x \in \mathbb{R}$ $\quad$ $A_{x \in \mathbb{R}} = \{(x,y) : y \in \mathbb{R}\}$

Union of $(\cup A_{x \in r}) = \cup\{(x,y) : y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R} = xy$-Plane

$(X.d)$

let $\{A_\alpha\}_{\alpha \in I}$ be the arbitrary collection in metric space $(X.d)$



1 Then if $\bigcup_{\alpha \in I} A_\alpha = \phi$ Then by define "ϕ" is open set in $(X.d)$

② If $\bigcup_{\alpha \in I} A_\alpha \neq \phi$, There exist $x \in A_\alpha$ for some $\alpha \in I$

A_α is open set in $(X.d)$, There exist (open sphere $r > 0$ at 'x') such That $S_r(x) \subseteq A_\alpha$ for some $\alpha \in I$

$S_r(x) \subseteq A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$ as x is arbitrary ~~function~~

$\bigcup_{\alpha \in I} A_\alpha$ is open set in $(X.d)$

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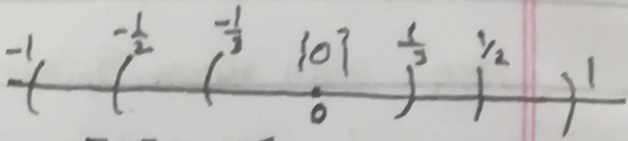
Land Mark Theorem for open sets (Finite intersection)

Finite intersection of open sets in a metric space is open set

Proof infinite



$$\bigcap_{n=1}^{\infty} \left]-\frac{1}{n}, \frac{1}{n}\right[= \left]-1,1\right[\cap \left]-\frac{1}{2}, \frac{1}{2}\right[\cap \left]-\frac{1}{3}, \frac{1}{3}\right[\cap \ldots\ldots \infty$$

If $\underset{n \to \infty}{} \bigcap_{n=\infty}^{\infty} \left]-\frac{1}{n}, \frac{1}{n}\right[= \{0\}$ It is singleton set is a close set

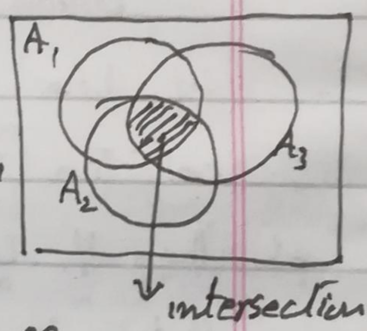
finite

let $\{A_1, A_2, \ldots A_n\} = \{A_\alpha\}_{\alpha=1}^{n}$ be The finite collection of open set

we are to prove $A_1 \cap A_2 \cap \cdots A_n = \bigcap_{\alpha=1}^{n} A_\alpha$ is open

Case I if $\bigcap_{\alpha=1}^{n} A_\alpha = \phi, X$, which is open sets

Case II



Let $\bigcap_{\alpha=1}^{n} A_\alpha \neq \phi$ Then

$x \in \bigcap_{\alpha=1}^{n} A_\alpha$, So, $x \in A_\alpha$, $\alpha = 1, 2, \ldots n$

$x \in A_1$, $x \in A_2 \ldots x \in A_\alpha$

A_α is open set $\forall \alpha = 1, 2, 3, \ldots n$

There exist $r_1, r_2 \ldots r_n > 0$ such That

$S_{r_1}(x) \subseteq A_1$, $S_{r_2}(x) \subseteq A_2 \ldots S_{r_n}(x) \subseteq A_n$

Let $r = \{\min\{r_1, r_2, \ldots r_n\}$

$S_r(x) \subseteq \{A_1, A_2, \ldots A_n\}$

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$S_r(x) \subseteq A_1 \cap A_2 \cap \ldots \cap A_n$

$S_r(x) \subseteq \bigcap_{\alpha=1}^{n} A_\alpha$, where x is arbitrary point

So, $\bigcap_{\alpha=1}^{n} A_\alpha$ finite Intersection is open set in metric space

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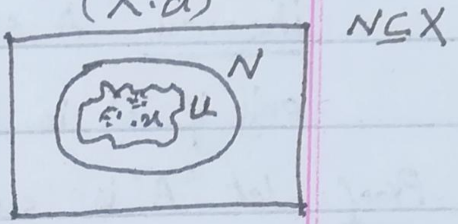
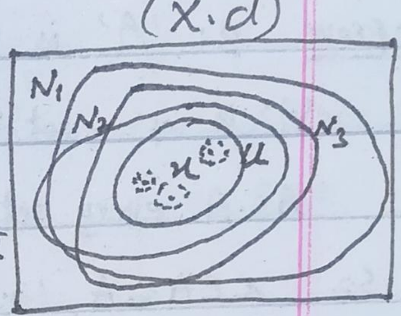
Neighbourhood of a point

Neighbourhood means subset: A subset 'N' is said to be The Neighbourhood of a poin 'x' If There exist an open set 'U' containing such That $x \in U \subseteq N$ stand for Nghd $(x) \to N_x$

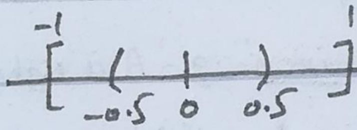


Neighbourhood system of point

If $x \in U \subseteq N_1, N_2, N_3 \ldots = \{N_i\}_{i \in I}$ is said to be Neighbourhood system of point 'x'



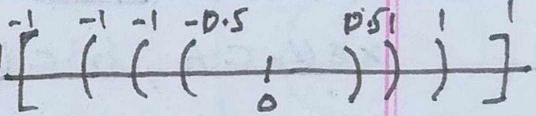
e.g on $\mathbb{R}$



$0 \in (-0.5, 0.5) \subseteq [-1,1]$

So, $[-1,1]$ is Nghd of 0

$0 \in (-0.5, 0.5) \subseteq [-1,1), (-1,1), (-1,1]$



are Nghd system of '0'

eg In discrete metric space every subset containg 'x' is Nghd of 'x' (x, d_0), we have $A \subseteq X$ such That $x \in A$, means $x \in A \subseteq X$ (X_x) Nghd of 'x'

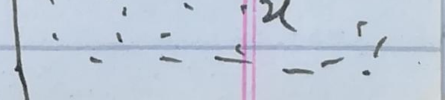
In (x, d_0), every singleton set is open $x \in \{x\} \subseteq A$

A is Nghd of 'x' $\forall x \in A$, A is Nghd of all its points.

Theorem the metric space itself is a Nghd of each of its point.

itself open
$(X.d)$

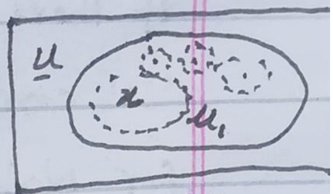
$x \in X \subseteq X$



Proof In a metric space $(X.d)$, let 'x' be an arbitrary point, Then $x \in X \subseteq X$, X is open set and X is also Nghd of 'x'

Theorem A subset of metric space is open if and only if it is the Neighbourhood of each of its point

Proof let 'A' is subset of 'X'



$x \in U_1 \subseteq U$

U_1 is open set

U is Nghd of x

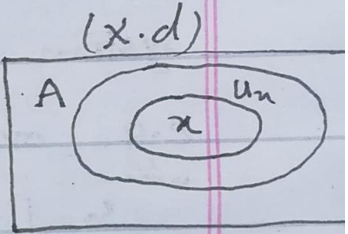
Given (i) p; A is an open set

To prove q: 'A' is Nghd of each of its point

'A' is open set in X, $\forall x \in A$ — I

$A \subseteq A$ every set is subset of itself — II

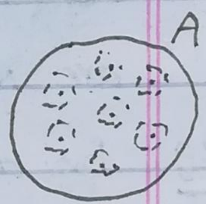
So, $x \in A \subseteq A$, 'x' is arbitrary point of 'A' & 'A' is Nghd of all its point.



Conversly **Given** : q : A is Nghd of each its point. **To proive** : 'A' is an open set

$x \in B$, $\{x\} \subseteq B$

$x \in U_x \subseteq A$, $\{x\} \subseteq U_x \subseteq A$



Taking Union for all such 'x'

$\bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x \subseteq A$; $\bigcup_{x \in A} U_x$ is union of open set

$$A \subseteq \bigcup_{x \in A} U_x \subseteq A$$

$A \subseteq B \subseteq A$, $A = B$

$$A = \bigcup_{x \in A} U_x$$

Hence A is open set, being arbitrary union of open set

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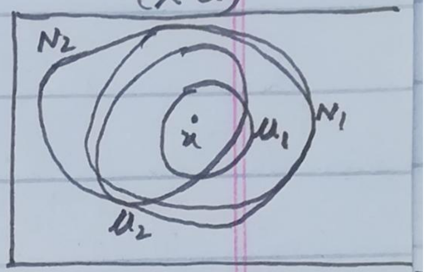
Theorem Finite intersection of the Neighbourhood of a point, is also its Neighbourhood.

(X.d)



Proof let $N_1, N_2, \ldots N_r = \{N_j\}_{j=1}^{r}$ be the Nghd system point 'x' in $(X.d)$

there exist $U_1, U_2, \ldots U_r = \{U_j\}_{j=1}^{r}$

$x \in U \subseteq N$

the collection of open sets

$x \in U_1 \subseteq N_1$, $x \in U_2 \subseteq N_2$, ... $x \in U_j \subseteq N_j$, $j = 1, 2, 3, \ldots r$

$x \in U_1 \cap U_2 \cap \ldots \cap U_r \subseteq N_1 \cap N_2 \cap, \ldots, \cap N_r$ | $x \in A$, $x \in B$

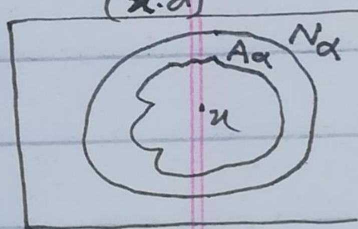
$x \in \bigcap_{j=1}^{r} U_j \subseteq \bigcap_{j=1}^{r} N_j$ | $x \in A \cap B$

$\bigcap_{j=1}^{r} U_j$ = finit intersection of open set so,

U_j = is an open set and $\bigcap_{j=1}^{r} N_j$ is Nghd of 'x'

Theorem Arbitrary union Neighbourhood of a point is also its Neighbourhood.

(x.d)



Proof let $\{N_\alpha\}_{\alpha \in I}$ be The arbitrary collection of Nghd of point 'x' in $(X.d)$

each N_α is Nghd of 'x' where $\forall \alpha \in I$

There exist an open set say A_α such that $x \in A_\alpha \subseteq N_\alpha$

$\forall \alpha \in I$ so, $x \in \bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{\alpha \in I} N_\alpha$ | $x \in A_1$

$\bigcup_{\alpha \in I} A_\alpha$ being arbitrary union of open sets. | $x \in A_1 \cup A_2 \cup \ldots$

So, $\bigcup_{\alpha \in I} N_\alpha$ is Nghd of 'x'

Divergent Sequence A sequence $\{x_n\}$ in a metric space $(X.d)$. However large $l>0$, There exist $n_0 \in N$ such That $n \geq n_0$. $\forall\ n \in N$, $a_n \notin S_l(l)$

e.g $\{a_n = n\}_{n=1}^{\infty}$ $n \in N$ $a_n \to \infty$

e.g $\{a_n = 5^n\}_{n=1}^{\infty}$ $n \in N$ $a_n \to \infty$

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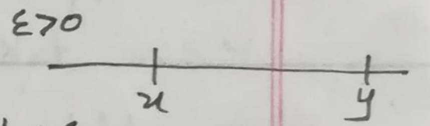
Uniqueness of a limit of a sequence in a metrice space.

__Theorem__ limit of a convergent sequence in a metric space is unique.

__Proof__ Let $\{x_n\} \to x$ and $\{x_n\} \to y$ in a metric space $(X.d)$ such That $x \neq y \Rightarrow d(x,y) \neq 0$

For any $\varepsilon > 0$ There exist $x \in N$ such That



$\dfrac{d(x,y)}{n} < \varepsilon$

$\dfrac{d(x,y)}{n} = 0 < \varepsilon$ $t < \varepsilon$, $t = 0$

it is not possible so, $x = y$

__Theorem__ Every convergent sequence in a metric space is bounded but converse may not true.

__Proof__ in $(X.d)$ let $\{x_n\}$ be a ~~metric space~~ a convergent sequence such That $x_n \to x \in N$

$\forall\ \varepsilon > 0$, there exist $n_0 \in N$ such That $\forall\, n \geq n_0$

$d(x_n, x_0) < \varepsilon$ — ①

let $K = \max\{d(x_1, x) + \varepsilon,\ d(x_2, x) + \varepsilon, \ldots d(x_{n-1}, x)\}$

$d(x_n, x) < K$, $\forall n \in N$, $\{x_n\}_{n=1}^{\infty} \subseteq S_K(x)$

$d(x_m, x_n) < K$, $\forall m, n \in N$

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So, $\{x_n\}_{n=1}^{\infty}$ is bounded sequence

conversly in (R, d)

$\{x_n\} = \{(-1)^n\}_{n=1}^{\infty}$ is a bounded sequence

Here $-1 \leq a_n \leq 1$ $\forall n \in N$

$a_n \to 1$ if n is even, $a_n \to -1$ if n is odd

limit of The sequence is not unique and

$\{x_n\}$ is not convergence

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convergence $\Rightarrow$ bounded

Bounded $\not\Rightarrow$ convergence.

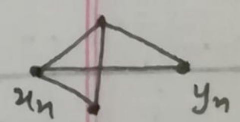
Theorem In a metric space $(X.d)$, if $\{x_n\} \to x$
$\{y_n\} \to y$ Then $d(x_n, y_n) \to d(x, y)$

Proof $\{x_n\} \to x$ $\forall \varepsilon > 0$, There exist $n_1 \in N$ such That $\forall n \geq n_1$, $d(x_n, x) < \varepsilon$ —— I

$\{y_n\} \to y$ $\forall \varepsilon > 0$, There exis $n_2 \in N$ such That $\forall n \geq n_2$ $d(y_n, y) < \varepsilon$ —— 2

Let $n_0 = \max\{n_1, n_2\}$

from 1, and 2
$$\left.\begin{array}{l} d(x_n, x) < \frac{\varepsilon}{2} \\ d(y_n, y) < \frac{\varepsilon}{2} \end{array}\right\} \forall n \geq n_0$$



$$|d(x_n, y_n) - d(x, y)| \leq |d(x_n, x) + d(x, y) + d(y, y_n) - d(x, y)|$$

$$\leq |d(x_n, x) + d(y_n, y)|$$

$$\le d(x_n, x) + d(y_n, y)$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} , = \frac{2\varepsilon}{2} = \varepsilon$$

$$|d(x_n, y_n) - d(x, y)| < \varepsilon \quad \forall \; n \ge n_0$$

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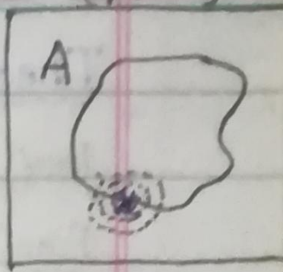
$$d(x_n, y_n) \longrightarrow d(x, y)$$

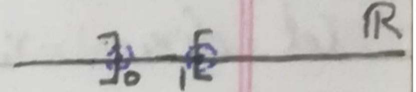
MTH 405 lecture 37 Nomi $(X \cdot d)$



Limit point of set In a metric space $(X \cdot d)$, Let $A \subseteq X$ and any point $x \in X$ is said to be the limit point of 'A' if every open sphere at 'x' contain at least one point of 'A' other than 'x'

$$S_r(x) \cap A - \{x\} \neq \phi \quad \text{where } r>0$$

e.g: 0, 1 are the limit points of



$]0,1[$ because any open interval at $0,1 \notin]0,1[$ '0' and '1' contain at least one point of $]0,1[$

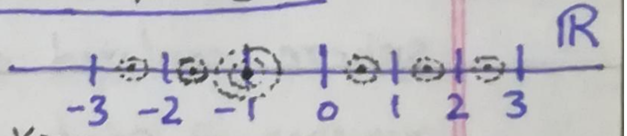
$S_r(0) \cap]0,1[- \{0\} \neq \phi$, $S_r(1) \cap]0,1[- \{1\} \neq \phi$

$(0-\varepsilon, 0+\varepsilon) \cap]0,1[- \{0\} \neq \phi$, $(1-\varepsilon, 1+\varepsilon) \cap]0,1[- \{1\} \neq \phi$

$(x-\varepsilon, x+\varepsilon) \cap]0,1[- \{x\} \neq \phi \quad \forall \varepsilon > 0, \forall x \in]0,1[$

e.g - Limit points of The set of integers $\mathbb{Z} \subseteq \mathbb{R}$

Sol $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ----\}$



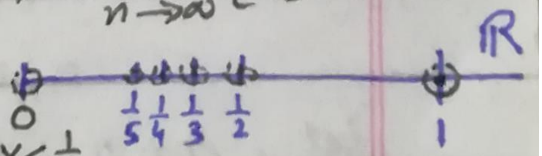
$\forall x \in \mathbb{R}$ There exist $r>0$

Such that $S_r(x) \cap \mathbb{Z} - \{x\} = \phi$

$\mathbb{Z}$ has no limit point because all integer's number's has some distance between each other

e.g; Limit point of $\{\frac{1}{n}\}_{n \in N}$; $\lim_{n \to \infty} [\frac{1}{\infty}] = \{0\}$

$\{\frac{1}{n}\}_{n \in N} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ----\}$



So, $\frac{1}{2} \notin S_{\frac{1}{2}}(1), r < \frac{1}{2}$

By induction $\frac{1}{n+1} \notin S_{\frac{1}{n+1}}(0)$ if $n \in N$ Then $n+1 \in N$

0, is a limit point of $\{\frac{1}{n}\}_{n \in N}$

Closed set / Derived set

A subset 'A' of metric space (X,d) $A \subseteq X$ is closed if it contains every limit point of itself.

The set of all limit points of A is called the derived set of A and denoted by A', A^c, A^d

e.g $0,1 \notin]0,1[$ $\quad A' = A^c = A^d =]0,1[^d = [0,1]$ is closed set

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}^d = \{\ \} = \phi$

$0 \notin \{\frac{1}{n}\}_{n \in N} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $\{\frac{1}{n}\}^d_{n \in N} = \{0\}$

Derived set of Rational (Q) and Irrational (Q')

Rational $(Q) = \{x = \frac{p}{q}, q \neq 0, p,q \in Z \subset R\}$

Irrational $(Q') = \{x \neq \frac{p}{q}, q \neq 0, p,q \in Z \subset R\}$

e.g $Q: \frac{1}{2}, 7, \frac{7}{2}$, e.g $Q' = \pi, \sqrt{29}, e$

Q is infinite on $\mathbb{R}$ line, Q' is infinite on $\mathbb{R}$ Line

either $x \in Q$ or $x \in Q'$

$S_r(x) \cap Q - \{x\} \neq \phi$ $\quad$ when $x \in Q$

$S_r(x) \cap Q' - \{x\} \neq \phi$ $\quad$ when $x \in Q'$

So, derived set of $= Q^d = (Q^c)^d = \mathbb{R}$

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Short Lecture
Virtual University 37,38 LIMIT POINT DERIVED SETS
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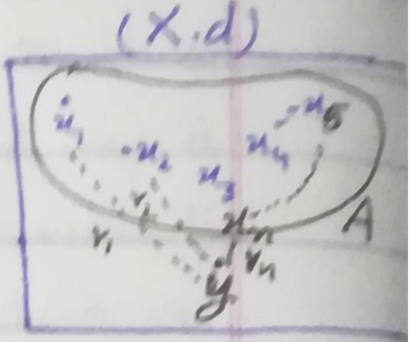
Theorem Finite subset of a metric space has no limit point.



Proof let $A \neq \phi$, $A \subseteq X$ be a subset of 'X' such That

$A = \{x_i\}_{i=1}^{n} = \{x_1, x_2, \dots x_n\}$

let y is Limit point of A, $S_{r_1}(y) \cap A - \{x\} = \{y\}$

Let $d(x_1, y) = r_1$, $d(x_2, y) = r_2 \dots\dots d(x_n, y) = r_n$

$r = \min\{r_1, r_2, \dots r_n\} = \min\{r_i\}_{i=1}^{n}$

$S_r(y) = \{y \in X \ \ d(y, x_n) < r_n\}$

So, $S_r(y)$ has no point of A, other Than 'y'

A has no limit point in 'X'

Theorem If 'x' is a limit point of a subset 'A' of a metric space $(X.d)$ Then every open sphere centered at 'x' contains infinite number of points of 'A' different from 'x'.

Theorem The derived set of a non-empty set of discrete metric space is empty.

Proof $d_0 : X_0 \times X_0 \longrightarrow \mathbb{R}$; $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

$A \subseteq X$, $A^d = \{x \in X ; x \text{ is limit point of } A\}$

$= \{x \in X ; S_r(x) \cap A - \{x\} \neq \phi\}$

Let, $A \neq \phi$, $A \subseteq X$ and $A^d \neq \phi$ There exist $x : x \in A^d$, x is a limit point of A

$\exists \ r > 0$ such that $S_r(x)$ should at least contain one point of 'A' other Than 'x'

if $r<1$ Then, $S_r(x)=\{y\in X_0 : d(y,x)<1\}$

$=\{y\in X_0 : d(y,x)=0\}$

$=\{x\} \quad y=x$

$x\notin A^d$, 'x' is not a limit point of 'A'

$A\cap S_r(x)-\{x\}=\phi$, Hence, $A^d=\phi$.

Set Theoretic operations in Derived sets:

For any non-empty subset 'A' and 'B' of a metric space $(X.d)$

(i) if $A\subseteq B$ Then $A^d\subseteq B^d$ (ii) $(A\cap B)^d\subseteq A^d\cap B^d$

(iii) $(A\cup B)^d=A^d\cup B^d$

Proof (i) $A^d=\{x\in X ; A\cap S_r(x)-\{x\}\neq\phi, \forall r>0\}$

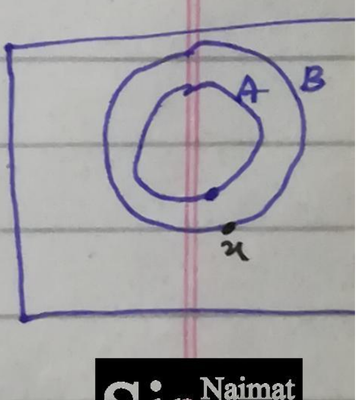
give $A\subseteq B$ and to prove $A^d\subseteq B^d$

Let $x\in A^d$; $A\cap S_r(x)-\{x\}\neq\phi \quad \forall\ r>0$ —— I

$B\cap S_r(x)-\{x\}\neq\phi \quad \because A\subseteq B$

So, x is a limit point of 'B' as well $x\in B^d$ —— II

So, $A^d\subseteq B^d$ from I and II



(ii) $(A\cap B)^d=A^d\cap B^d$

$A\cap B\subseteq A \quad \wedge, \quad A\cap B\subseteq B$

$(A\cap B)^d\subseteq A^d \quad \wedge, \quad (A\cap B)^d\subseteq B^d$

$(A\cap B)^d\subseteq A^d\cap B^d$

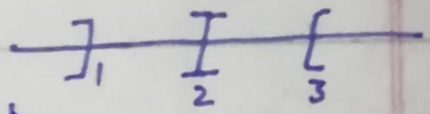
Counter example $\quad A^d\cap B^d=]1,2[$

$(A\cap B)^d=]2,3[$

$]1,2[^d = [1,2]$, $]2,3[^d = [2,3]$

$]1,2[^d \cap]2,3[^d = [1,2] \cap [2,3] = \{2\}$

$]1,2[\cap]2,3[= \phi$



$(]1,2[\cap]2,3[)^d = \phi^d = \phi$

$\phi \neq \{2\}$

$Q \cap Q^c = \phi$

$\mathbb{R} \cap \mathbb{R} = \mathbb{R}$ $\quad \phi \neq \mathbb{R}$

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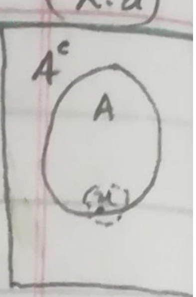


<u>Closed set</u> A subset 'A' of a metric space $(X.d)$, $A \subseteq X$ is closed set if it contains every limit point of itself

e.g 'o' is The limit point of $\{\frac{1}{n}\}_{n=1}^{\infty}$ but 'o' $\notin \{\frac{1}{n}\}_{n=1}^{\infty}$

So, it is not closed set.

e.g 'o' and '1' are The limit point of open Interval $]0,1[$ but $0,1 \notin]0,1[$ is not closed set.

e.g 'o' and '1' are The limit point of close interval $[0,1]$ and $0,1 \in [0,1]$ is closed set.

e.g $R = Q \cup Q^c$, $Q^d = R = Q \cup Q^c$, $(Q^c)^d = R = Q \cup Q^c$

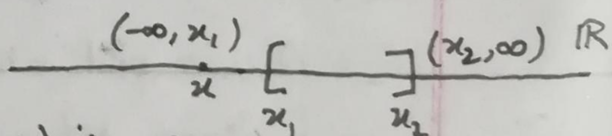
if $x \in Q^c$, Then its limit point is Q and $x \notin Q$

if $x \in Q$, Then its limit point is Q^c and $x \notin Q^c$

So, Q and Q^c are not closed in $\mathbb{R}$

<u>Example</u> close intereval on $\mathbb{R}$ line are closed set

$[x_1, x_2]^c = (-\infty, x_1) \cup (x_2, \infty)$



we porove $(-\infty, x_1) \cup (x_2, \infty)$ is open

Let $x \in (-\infty, x_1) \cup (x_2, \infty)$, $x \in (-\infty, x_1)$ or $x \in (x_2, \infty)$

if $x \in (-\infty, x_1)$ and taking $r = |x_1 - x| > 0$ such That

$S_r(x) = \{y \in \mathbb{R} : d(x,y) < r\} =]x, y[\subseteq (-\infty, x_1)$

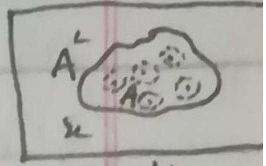
$S_r(x) \subseteq (-\infty, x_1)$, $S_r(x) \subseteq (-\infty, x_1) \cup (x_2, \infty)$

So, $(-\infty, x_1) \cup (x_2, \infty)$ is open set

$[(-\infty, x_1) \cup (x_2, \infty)]^c = [x_1, x_2]$ is closed set.

Example every subset of discrete metric space is closed set.

$$d_0(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$



$(X \cdot d_0)$

let $\phi \neq A \subseteq (X, d_0)$, To prove 'A' is closed set and 'A^c' is open set

let $x \in A^c$, for $r = 1$, Taking open sphere $r = 1$ at

$x \in A^c$, $S_r(x) = \{y \in X, \ d_0(x,y) < 1\}$

$= \{y \in X, \ d_0(x,y) = 0\}$

$= \{x\}, \quad x = y$

$\{x\} \subseteq A^c$, $S_r(x) \subseteq A^c$ as 'x' is arbitrary point of A^c, so, A^c is open set, $(A^c)^c = A$ is closed set

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Theorem The union of finite metric / number of closed sets is closed.

Proof Finite collection of closed sets $= \{A_1, A_2, \dots A_n\} = \{A_i\}_{i=1}^{n}$

$\{A_1^c, A_2^c, \dots A_n^c\} = \{A_i^c\}_{i=1}^{n}$ is open set | Finite intersection of open set is open

$A_1^c \cap A_2^c, \dots \cap A_n^c\} = \bigcap_{i=1}^{n} A_i^c$ is open sets

$\{A^c \cap B^c = (A \cup B)^c$ de margon's Law

$\{A_1^c \cap A_2^c \dots \cap A_n^c\} = \bigcap_{i=1}^{n} A_i^c = (\{A_1 \cup A_2 \cup \dots \cup A_n\})^c = \left(\bigcup_{i=1}^{n} A_i\right)^c$

is open, Then $\left[\left(\bigcup_{i=1}^{n} A_i\right)^c\right]^c = \bigcup_{i=1}^{n} A_i$ is closed set

Infinitely

Let $A_n = [\frac{1}{n}, 1]$

$A_1 = [1, 1]$, $A_2 = [\frac{1}{2}, 1]$, ...

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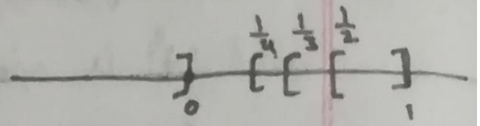
as $n \to \infty$ Then $\frac{1}{n} = 0$, $A_n = (0,1]$ as $n \to \infty$

$A_1 \cup A_2 \cup \ldots . A_\infty = (0,1]$ is not closed sets



Theorem the Union of arbitrary number of closed sets is closed.

Theorem The intersection of finite number of closed set is closed.

Closeness of a finite subset of 'A' metric space.

Theorem ; a finite subset of a metric space is closed set.

Proof Let $\phi \neq A \subseteq X$ be a non-empty subset of X

P ; 'A' is finite q ; 'A' is closed

$P \to q$ = اگر زہر کھائیں گے تو مر جائیں گے

$P \to q \cong \sim q \to \sim P$

$\sim q \to \sim P$ = مریں گے نہیں اگر زہر نہیں کھائیں گے

$\sim q$; 'A' is not closed set

There exist $x \in X$: $x \in A^d$ but $x \notin A$

$x \in A^d$, 'x' is a limit point of 'A'

$\sim P$ = 'A' is infinite
$\sim q$ = 'A' is not closed set

every open sphere at 'x' will contain infinite many points of 'A'

So, 'A' is infinit ($\sim P$)

So, $\sim q \to \sim P = P \to q$

Theorem In a metric space a subset is closed set if and only if its compliment is open.

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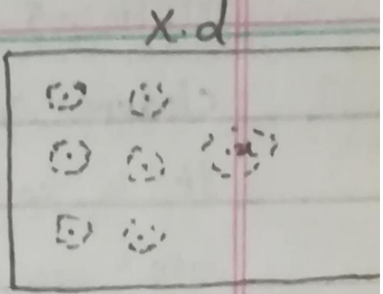


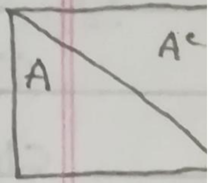
Proof Let $(X.d)$ be a metric space

(i) $\phi \subseteq X$, ϕ is finite, ϕ is closed

$\phi^c = X$ is open

(ii) $X \subseteq X$, X is closed set as $X^d = X$

$X^c = \phi$ is open trivially.



(iii) Let $A \subseteq X$ such that $A \neq \phi$, $A = X$

Suppose 'A' is closed (Given)

A^c is open (to prove)

Let $x \in A^c$ Then $x \notin A$, so, A is closed set $A = A^d$

$x \notin A^d$; 'x' is not a limit point of 'A' Then

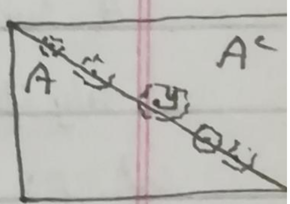
There exist $r > 0$ such That $A \cap S_r(x) = \phi$

$S_r(x) \subseteq A^c$, $\forall\, x \in A^c$, as 'x' is arbitrary point

A^c is open set.

Conversly Let A^c is open (Given) to prove A^c is closed

Let ~~ϕ~~ A is not closed, $A \neq A^d$



$y \in X$ such That $y \in A^d$ but $y \notin A$

if $y \in A^d$ Then 'y' is limit point

of 'A' Taking a open sphere at 'y' $\forall\, l > 0$

$S_l(y) \cap A \neq \phi$ — I

$y \notin A$ Then $y \in A^c$, A^c is open set, There exist

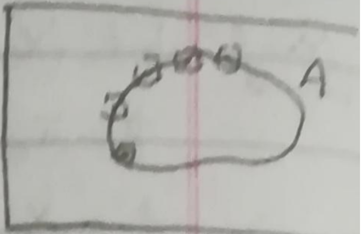
$p > 0$ such That $S_p(y) \subseteq A^c$, $S_p(y) \notin A$ Then

$S_p(y) \cap A = \phi$ — II so, our contradiction

is wrong.

Closure of a set in metric space.

It is collection of all Those points which are arbitrary closed to "A"



Closure of $A = \bar{A} = A \cup A^d$

e.g $Q^d = R$, $\bar{Q} = Q \cup Q^d = Q \cup \mathbb{R} = \mathbb{R}$

$(Q^c)^d = R$, $\overline{Q^c} = Q^c \cup (Q^c)^d = Q^c \cup R = \mathbb{R}$

Closure of Harmonic Sequence

$A = \{\frac{1}{n}\}_{n \in N} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, $A^d = (\{\frac{1}{n}\}_{n \in N})^d = \{0\}$

$\bar{A} = A \cup A^d = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Closure of an open set in $\mathbb{R}$

$A =]0,1[$, $A^d = (0,1)$; $0,1 \notin \overset{\text{open}}{]0,1[}$, $0,1 \in \overset{\text{close}}{[0,1]}$

$\bar{A} = A \cup A^d = (0,1) \cup (0,1)^d = [0,1]$

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Closeness is a closure Theorem:

A subset 'F' of a metric space, is closed if and only if $F = \bar{F}$

Let p: F is a closed set , q; $F = \bar{F} = F \cup F^d$

if 'F' is closed, Then 'F' contain all of its $\overset{\text{limit}}{\text{point}}$

$F^d \subseteq F$ — I, $F \cup F^d = \bar{F} \cup F$, $\bar{F} \subset F$ — II

By Defination $\bar{F} = F \cup F^d$

$F \subseteq \bar{F}$ — III. from II and III

$\bar{F} \subseteq F \subseteq \bar{F}$, $F = \bar{F}$

$A \subseteq A \cup B$

$A \subseteq B \subseteq A$

$A = B$

conversly

Let $F = \bar{F}$ (given) to prove q; F is closed sets

$(\sim q \to \sim p)$, $\sim q$ = F is not closed set Then

There exist a limit points say 'p' of F such

That $p \notin F$ Then $p \notin \bar{F}$ because $F = \bar{F}$

$p \notin F \cup F^d$ $\because$ $\bar{F} = F \cup F^d$, $p \notin F^d$ so,

p is not a limit point of 'F'

contradiction is wrong so, F is closed

Set Theoretic operation in closures.

Theorem; If 'A', and 'B' are two subset of a metric space $(X.d)$, Then prove that

(i) $A \subseteq B$ Then $\bar{A} \subseteq \bar{B}$ (ii) $(\overline{A \cup B}) = \bar{A} \cup \bar{B}$

(iii) $(\overline{A \cap B}) = \bar{A} \cap \bar{B}$ (iv) $\bar{A}$ is ~~largest~~ largest closed super set of A

Proof (i) given $A \subseteq B$ Then $A^d \subseteq B^d$ taking a union

$A \cup A^d \subseteq B \cup B^d$ Then $\bar{A} \subseteq \bar{B}$

(ii) $L.H.S = \overline{A \cup B}$

$= (A \cup B) \cup (A \cup B)^d$ $\qquad \bar{A} = A \cup A^d$

$= (A \cup B) \cup (A^d \cup B^d)$

$= (A \cup A^d) \cup (B \cup B^d)$

$= \bar{A} \cup \bar{B}$

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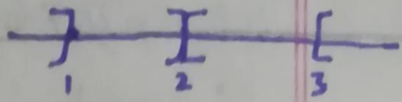
(iii) $(\overline{A \cap B}) = \bar{A} \cap \bar{B}$, $A \cap B \subseteq A$ and $A \cap B \subseteq B$

$\overline{A \cap B} \subseteq \bar{A}$ and $\overline{A \cap B} \subseteq \bar{B}$

taking intersection; $(\overline{A \cap B}) \cap (\overline{A \cap B}) \subset \bar{A} \cap \bar{B}$

$(\overline{A\cap B}) \subseteq \bar{A}\cap\bar{B}$

e.g let $A=(1,2)$, $B=(2,3)$



$\bar{A}=[1,2]$ $\bar{B}=[2,3]$

L.H.S = $(\overline{A\cap B}) = \overline{(1,2)\cap(2,3)} = \bar{\phi} = \phi$ $\because F=\bar{F}$

R.H.S = $\bar{A}\cap\bar{B} = [1,2]\cap[2,3] = \{2\}$

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so, $\phi \subseteq \{2\}$

$(\overline{A\cap B}) \subseteq \bar{A}\cap\bar{B}$

IV For any $\phi \neq A \subseteq X$, $\bar{A}$ is closed

$\bar{A} = A\cup A^d$, $A\subseteq\bar{A}$ — I

Let $\phi\neq B\subseteq X$, be arbitrary any other closed set containing 'A' , $A\subseteq B$, $\bar{A}\subseteq\bar{B}$, $\bar{A}\subseteq B$ — II

$\bar{B}=B$

from I and II $A\subseteq\bar{A}\subseteq B$

$\bar{A}$ is smallest closed super set of A

Let $\phi\neq B\subseteq X$ be any other arbitrary closed set contain in 'A' $B\subseteq A$ Then $\bar{B}\subseteq\bar{A}$, $B\subseteq\bar{A}$ — III

from I, III $B\subseteq A\subseteq\bar{A}$, $\bar{A}$ is largest closed super set of A

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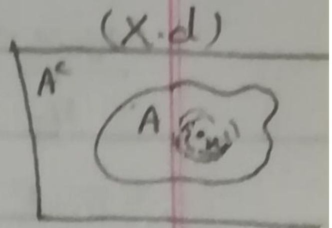
Interior, exterior and Boundary of a set in metric space (X.d)



Interior $x \in S_r(x) \subseteq A$

$Int(A)=\{x\in A, A\subseteq X, S_r(x)\subseteq A$ for some $r>0\}$

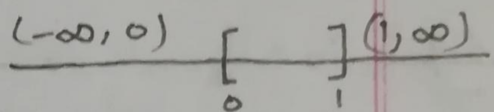
exterior $Ext(A)=\{x\in A^c; A^c\subseteq X, S_r(x)\subseteq A^c$ for some $r>0\}$

$ext(A)=Int(A^c)$

Boundary $Fr(A)=\{x\in X; Int(A)\cap S_r(x)\neq\phi,$

Frontier $ext(A)\cap S_r(x)\neq\phi, \forall r>0\}$

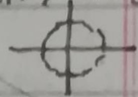
e.g $A=[0,1]$



$Int(A)=(0,1)$, $ext(A)=(-\infty,0)\cup(1,\infty)$

$Fr(A)=\{0,1\}$

$A=\{(x,y)\in R^2, x^2+y^2=1\}$



e.g $Int\{1\}\neq\phi$, $Int\, Z=\phi$

$Int(A), x^2+y^2<1$

e.g Discrete metric space (X.d)

Say $A\subseteq X$, $Int(A)=\bigcup_{x\in A}\{x\}=A$

$ext(A), x^2+y^2>1$

$ext(A)=\bigcup_{x\in A^c}\{x\}=A^c$

$Fr(A), x^2+y^2=1$

$Fr(A)=\phi$

Theorem $Int(A)\cup ext(A)\cup Fr(A)=X$

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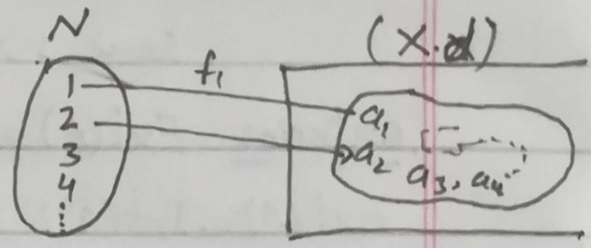
Sequences in a metric space.

<u>Sequence</u> infinit list of numbers.

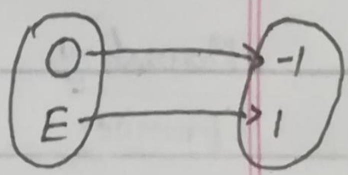
" It is a function whose domain is set of Natural numbers and whose range is subset of a metric space



$\{a_n\}_{n=1}^{\infty}$, $(a_n) = \{a_1, a_2, \ldots\}$

$\{a_n\}$, $a_n \rightarrow$ nth term

<u>e.g</u> $f(n) = (-1)^n \quad n \in N$



$f(1) = -1$, $f(2) = 1 \ldots\ldots$

<u>e.g</u> $a_n = \frac{1}{n}$, $a_n = n^2$

Montonicity of sequences.

$a_n = \frac{1}{n}$

(i) increasing squences $a_{n+1} \geq a_n \quad \forall\, n \in N$

ii) Decreasing sequences $a_{n+1} \leq a_n \quad \forall\, n \in N$

<u>Constant sequences.</u>

$a_n = n^2$

It is not increasing and decreasing e.g $a_n = 3$

$a_n = a_\infty$ = No such term exist $\infty \notin N$ $\quad n \in N$

<u>Bounded sequences</u> let $(X.d)$ be a metric space an $\{x_n\}$ be a sequence in X, Then its is said to be bounded if There exist $m \in \mathbb{R}$ $m < \infty$ such That $d(x_m, x_n) < M \quad \forall\, m, n \in N$

e.g $d: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} : d(x,y) = |x-y|, \ \forall\, x, y \in \mathbb{R}$

$\left\{x_n = \frac{1}{n}\right\}_{n \in N}^{\infty} : d(x_m, x_n) = \left|\frac{1}{m} - \frac{1}{n}\right| < 1$

it is bounded sequence

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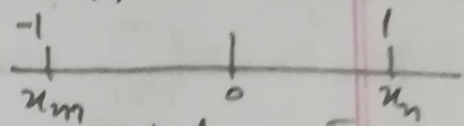
e.g $\{a_n = (-1)^n\}_{n=1}^{\infty}$, $n \in N$ is bounded sequences

because $-1 \le a_n \le 1 \quad \forall\, n \in N$

-1 is lower bound and 1 is upper bound.

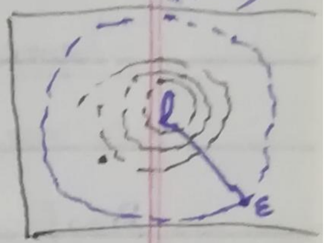
$d(x_m, x_n) \le 2$



e.g $\{a_n = n\}_{n=1}^{\infty}$ $n \in N$ it is unbounded as There not exist any $K \in \mathbb{R}$ such That $|a_n - a_m| < K$

Convergent Sequence in a metric space.

A sequence $\{x_n\}$ in a metric space $(X.d)$ is said to be convergence to limit 'l' if $\forall$ open spheres at 'l' There exist $n_0 \in N$ such That $x_n \in S_\varepsilon(l)$ $\forall\, n \ge n_0$



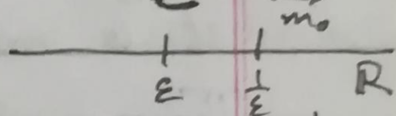
Symbolically $\{x_n\} \to l$, $x_n \to l$ $\lim\limits_{n\to\infty} x_n = l$

e.g $\{a_n = \frac{1}{n}\}$ Show That $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\forall\, \varepsilon > 0$ taking $|a_n - 0| < \varepsilon$, $|\frac{1}{n} - 0| < \varepsilon$, $\frac{1}{n} < \varepsilon$

$n > 0$, $|\frac{1}{n}| = \frac{1}{n}$, So, $\frac{1}{n} < \varepsilon$, $n > \frac{1}{\varepsilon}$

if m_0 is such a natural no



Just $\ge \frac{1}{\varepsilon}$, Then $|a_n - 0| < \varepsilon$, $\forall\, n \ge m_0$ Just $\ge \frac{1}{\varepsilon}$

$a_n \to 0$

e.g $\{\frac{1}{5^n}\} \to 0$ let the given $\varepsilon \ge 0$, taking $|a_n - 0| < \varepsilon$

$|\frac{1}{5^n} - 0| < \varepsilon$, $\frac{1}{5^n} < \varepsilon$, $5^n < \frac{1}{\varepsilon}$

$\log_5 5^n \le \log_5(\frac{1}{\varepsilon})$, $n \log_5 5 \le \log_5(\frac{1}{\varepsilon})$

$n < \log_5(\frac{1}{\varepsilon})$; if "$m_0$" is a natural no

Just $\ge \log_5(\frac{1}{\varepsilon})$ Then $\{\frac{1}{5^n}\} \to 0$